

More toposes and C^* -algebras

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Introduction 1

A unital C^* -algebra $\mathcal{A}$ has associated with it a category $\mathcal{M}(\mathcal{A}) = \mathcal{M}$, which I call the division category of (the underlying ring of) $\mathcal{A}$. A typical object of $\mathcal{M}$ is an element of $\mathcal{A}$, denoted R, S etc., and a morphism $U : R \longrightarrow S$ is an element U of $\mathcal{A}$ such that $\text{Ann}(R) = \text{Ann}(U)$ and $U = SX$ for some X , where $\text{Ann}(R) = \{X \in \mathcal{A} \mid RX = 0\}$. (There is also a category with the same objects but with morphisms U such that $\text{Ann}(R) \subseteq \text{Ann}(U)$; however, I prefer to work with $\mathcal{M}$ as it offers a bit more flexibility.) For instance, for any element R of $\mathcal{A}$, the objects R and R^*R are isomorphic in $\mathcal{M}$. It is sometimes good to know that there is a weak equivalence functor from $\mathcal{M}$ to the category of principal right ideals of $\mathcal{A}$ and injective $\mathcal{A}$ -module maps between them mapping an object R of $\mathcal{M}$ to the right ideal $R\mathcal{A}$ it generates. Let $\mathcal{M}$ denote the topos of presheaves on $\mathcal{M}$.

Introduction 2

We shall discuss three aspects beginning with:

1. *The positive quotient and polar decomposition.* An operator T on Hilbert space has a polar form $T = UA$, where A is positive, U is a partial isometry, and $\text{Ann}(T) = \text{Ann}(U)$. Let us say that a C^* -algebra $\mathcal{A}$ admits polar decomposition if every element of it has such a decomposition. It follows that if $\mathcal{A}$ admits polar decomposition, then necessarily $\mathcal{A}$ is supported as I call it. (Moreover, it follows that $A = |T| = \sqrt{T^*T}$ and that the decomposition is unique.) How do we generalize polar decomposition to all C^* -algebras, and not just the supported ones? This question has an answer in terms of a canonical correspondence between wide étale subcategories of $\mathcal{M}$ and the quotients in $\mathcal{M}$ of the representable presheaf associated with unit of $\mathcal{A}$.
2. *The Gelfand spectrum of a commutative C^* -algebra:*
1-dimensional representations. How this is related to $\mathcal{M}$ follows a pattern similar to the Zariski spectrum of a commutative ring.

Introduction 3

3. *GNS-representation theory (in any dimension), and the spectrum of an arbitrary C^* -algebra.* One of our tools is what we shall call a *seminormed $\mathcal{A}$ -module*, by which we mean a right $\mathcal{A}$ -module $\mathcal{V}$ that carries a seminorm such that

$$\forall v \in \mathcal{V}, \forall U \in \mathcal{A}, \|vU\| \leq \|v\| \|U\|.$$

(Thus, if $\|v\| = 0$, then $\|vU\| = 0$ for every U .) The functional dual $B(\mathcal{A})$, consisting of bounded $\mathbb{C}$ -linear maps $\tau : \mathcal{A} \rightarrow \mathbb{C}$, is an important $\mathcal{A}$ -module especially for GNS-theory. Its right action is defined by

$$\tau U(X) = \tau(XU^*); \tau \in B(\mathcal{A}). \quad (1)$$

What is more this action satisfies $\|\tau U\| \leq \|\tau\| \|U\|$, so $B(\mathcal{A})$ is a normed $\mathcal{A}$ -module. $B(\mathcal{A})$ classifies functionals in the sense that for any seminormed $\mathcal{A}$ -module $\mathcal{V}$ there is a natural isomorphism

$$B(\mathcal{V}) \cong \mathcal{A}\text{-Bdd}(\mathcal{V}, B(\mathcal{A})). \quad (2)$$

Introduction 4

Another important tool is how to parameterize a (right) $\mathcal{A}$ -module as an object of $\mathcal{M}$: if $\mathcal{V}$ is an $\mathcal{A}$ -module, then

$$\widehat{\mathcal{V}}(R) = \mathcal{A}\text{-Mod}(R\mathcal{A}, \mathcal{V}) \cong \{ v \in \mathcal{V} \mid \text{Ann}(R) \subseteq \text{Ann}(v) \}$$

is a presheaf on $\mathcal{M}$. If $\mathcal{V}$ is a seminormed $\mathcal{A}$ -module, then an $\mathcal{A}$ -module map $R\mathcal{A} \longrightarrow \mathcal{V}$ is necessarily bounded. Therefore, we have

$$\widehat{\mathcal{V}}(R) = \mathcal{A}\text{-Bdd}(R\mathcal{A}, \mathcal{V}) \text{ (bounded } \mathcal{A}\text{-module maps)}. \quad (3)$$

The presheaf associated with $B(\mathcal{A})$ is given by

$$\widehat{B(\mathcal{A})}(R) \overset{(3)}{=} \mathcal{A}\text{-Bdd}(R\mathcal{A}, B(\mathcal{A})) \overset{(2)}{\cong} B(R\mathcal{A}). \quad (4)$$

Introduction 5

From this point of view transition in $\widehat{B(\mathcal{A})}$ along a morphism $U : R \longrightarrow S$ of $\mathcal{M}$ is achieved in the following way by passing to principal right ideals.

$$\begin{array}{ccc}
 RA & \xrightarrow{RX \mapsto UX} & SA \\
 \searrow \tau \cdot U & & \downarrow \tau \\
 & & \mathbb{C}
 \end{array} \quad \tau \cdot U(RX) = \tau(UX)$$

The upshot is that $\widehat{B(\mathcal{A})}$ is the presheaf of functional germs on $\mathcal{A}$, suggesting that $\widehat{B(\mathcal{A})}$ may be interpreted as the complex numbers object of the topos $\mathcal{M}$. In any case, $\widehat{B(\mathcal{A})}$ is a ring object with a conjugation operation (such that $\overline{\tau(SX)} = \tau(SX)$) internal to $\mathcal{M}$.

Supported C^* -algebra 1

Support/cosupport projection

- A support projection $C(T)$ satisfies $C(T) \leq P$ iff $T = PT$
- A cosupport projection $N(T)$ satisfies $P \leq N(T)$ iff $TP = 0$

Support hypothesis

We shall say that a C^* -algebra $\mathcal{A}$ is *supported* if:

- 1 every $T \in \mathcal{A}$ has a support projection $C(T)$ such that
- 2 $\forall S, T, R : C(S) \leq C(T) \Rightarrow C(RS) \leq C(RT)$ (stability).

Equivalent cosupport hypothesis

- 1 every T has a cosupport projection $N(T)$ such that
- 2 $\forall S, T, R : N(S) \leq N(T) \Rightarrow N(SR) \leq N(TR)$.

Supported C^* -algebra 2

For any $T \in \mathcal{A}$ let $\text{Ann}(T) = \{X \in \mathcal{A} \mid TX = 0\}$

Blackadar [1]

For every $T \in \mathcal{A}$ there is a projection P such that $\text{Ann}(T) = \text{Ann}(P)$. This implies $P = C(T^*)$.

Rickart

For every $T \in \mathcal{A}$ there is an idempotent E such that $\text{Ann}(T) = \text{Ann}(E)$.

Theorem

The three notions supported, Blackadar, and Rickart are equivalent.

Kerzman-Stein formula [2]: $C(E) = E(E + E^* - I)^{-1}$

von Neumann algebra

$B(\mathcal{H})$ and more generally any von Neumann algebra is supported in this sense.

$C(X)$

What about the C^* -algebra $C(X)$ of continuous functions on a (connected) compact Hausdorff space X ?

Generally, $C(X)$ is not supported.

The division category 1

Definition of $\mathcal{M}(\mathcal{A}) = \mathcal{M}$

Objects: elements of $\mathcal{A}$

Morphisms: $U : R \longrightarrow S$ such that $\text{Ann}(R) = \text{Ann}(U)$, and $U \in S\mathcal{A}$ (also written $S \mid U$).

$$\begin{array}{ccc} R \xrightarrow{U} S & R \xrightarrow{R} R & R \xrightarrow{U} S \\ \searrow \frac{VU}{S} & \searrow \frac{UR}{R} = U & \searrow \frac{SU}{S} = U \\ & \downarrow V & \downarrow U \\ & T & S \end{array}$$

If $U = SX$, then $\frac{VU}{S} = VX$.

The division category 2

The divisor preorder

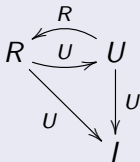
Let $\text{DIV}(\mathcal{A})$ denote the divisor preorder $\preceq$ on (the underlying ring of) $\mathcal{A}$ such that $U \preceq R$ if $U = RX$ for some X .

Proposition

Then $\text{DIV}(\mathcal{A}) \simeq \mathcal{M}/I$.

Proof

Objects: R maps to $R : R \longrightarrow I$, and in the other direction $U : R \longrightarrow I$ to U . The top morphism is an isomorphism.



The division category 3

Definition

Let $\mathcal{M}$ denote the topos of presheaves on $\mathcal{M}$.

Representable presheaf

Let $S \in \mathcal{A}$. $\widehat{S} : \mathcal{M}^{\text{op}} \longrightarrow \text{Set}$

$\widehat{S}(R) = \mathcal{M}(R, S) = \{ \text{morphisms } U : R \longrightarrow S \text{ of } \mathcal{M} \}$

$= \{ U \mid \text{Ann}(R) = \text{Ann}(U) \text{ and } S \mid U \}.$

Representable presheaf associated with the unit I

$\widehat{I} : \mathcal{M}^{\text{op}} \longrightarrow \text{Set}$

$\widehat{I}(R) = \{ U \in \mathcal{A} \mid \text{Ann}(R) = \text{Ann}(U) \}$

Note: $R \in \widehat{I}(R)$ so $\widehat{I}$ has global support.

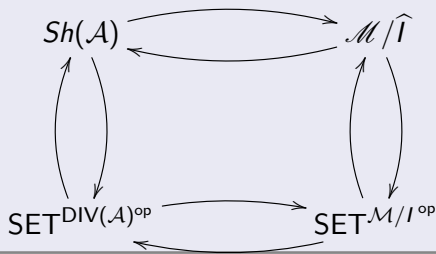
The division category 4

Proposition

$\mathcal{M}$ is an étendue.

Proof

A right $\mathcal{A}$ -subset of $\mathcal{A}$ is the same as a downset of $\text{DIV}(\mathcal{A})$. Let $\mathcal{O}(\mathcal{A})$ denote the frame of right $\mathcal{A}$ -subsets of $\mathcal{A}$, with topoi $\text{Sh}(\mathcal{A})$. $\mathcal{O}(\mathcal{A})$ and the frame $\text{Sub}(\hat{I})$ are isomorphic.



Polar decomposition 1

Definition

A subcategory $\mathcal{D} \longrightarrow \mathcal{M}$ is an *étale subcategory* if:

① for any two objects R, S of $\mathcal{D}$, if $S \mid R$, then the morphism $R : R \longrightarrow S$ is in $\mathcal{D}$;

② in a triangle $R \xrightarrow{U} S$ of composable morphisms of $\mathcal{M}$

$$\begin{array}{ccc} R & \xrightarrow{U} & S \\ & \searrow & \downarrow V \\ & \frac{VU}{S} & T \end{array}$$

if $V, \frac{VU}{S} \in \mathcal{D}$, then $U \in \mathcal{D}$.

Example: projections and partial isometries

Let $\partial \longrightarrow \mathcal{M}$ denote the subcategory whose objects are all the projections of $\mathcal{A}$, such that a morphism $U : P \longrightarrow Q$ is in ∂ just when U is a partial isometry.

Polar decomposition 2

Proposition

A morphism $U : P \longrightarrow Q$ of $\mathcal{M}$ is in ∂ iff $P = U^*U$.
 $\partial \longrightarrow \mathcal{M}$ is an étale subcategory.

Proof

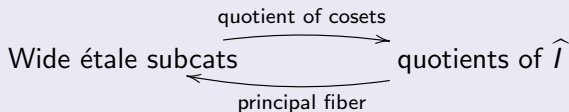
If U is a partial isometry, then it follows that $P = C(U^*) = U^*U$.
Conversely, if $P = U^*U$, then U is a partial isometry.

Definition

A subcategory $\mathcal{D} \longrightarrow \mathcal{M}$ is *wide* if every object of $\mathcal{M}$ is in $\mathcal{D}$.

Theorem

Wide étale subcategories of $\mathcal{M}$ and quotients of $\hat{I}$ correspond.



Example

Let $\mathcal{I} \subseteq \mathcal{A}$ be a right ideal. Then the morphisms $U : R \longrightarrow S$ such that $U \equiv R \pmod{\mathcal{I}}$ form a wide étale subcategory of $\mathcal{M}$.

Lemma 1

Suppose $\mathcal{D} \longrightarrow \mathcal{M}$ is a wide étale subcategory. Suppose we are given S, V such that $\text{Ann}(S) = \text{Ann}(V)$. Then t.f.a.e.

- 1 the morphism $V : S \longrightarrow I$ is in $\mathcal{D}$;
- 2 for any T , if $T \mid V$, then $V : S \longrightarrow T$ is in $\mathcal{D}$;
- 3 the morphism $V : S \longrightarrow V$ is in $\mathcal{D}$.

Lemma 2

A wide étale subcategory of $\mathcal{M}$ is closed under domain restrictions.

Proof

The domain restriction of a morphism is the following factorization. Let $U = \frac{VR}{S}$.

$$\begin{array}{ccc} R & \xrightarrow{U} & U \\ R \downarrow & & \downarrow U \\ S & \xrightarrow{V} & T \end{array}$$

If the morphism V is in $\mathcal{D}$, then $R \xrightarrow{U} U$ is in $\mathcal{D}$.

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Coset of a subcategory $\mathcal{D} \longrightarrow \mathcal{M}$

Let $U \in \mathcal{A}$. Let

$$\mathcal{D}U = \left\{ \frac{VU}{S} \mid S \mid U \text{ and } V : S \longrightarrow T \in \mathcal{D} \right\}$$

We call $\mathcal{D}U$ a coset of $\mathcal{D}$. $\mathcal{D}U$ is a subset of $\mathcal{A}$.

Example

$$\mathcal{D}I = \left\{ \frac{V}{S} \mid S \mid I \text{ and } V : S \longrightarrow T \in \mathcal{D} \right\}$$

Polar decomposition 7

Lemma 3

Suppose $\mathcal{D} \longrightarrow \mathcal{M}$ is a subcategory.

- 1 If a morphism $V : S \longrightarrow T$ of $\mathcal{M}$ is in $\mathcal{D}$, then $V \in \mathcal{D}S$.
- 2 $U \in \mathcal{D}U$ iff there is an object S of $\mathcal{D}$ such that $S \mid U$.

Proof

1. Use $S \xrightarrow{S} S \xrightarrow{V} T$ equals V .
2. Use $U \xrightarrow{U} S \xrightarrow{S} S$ equals U .

Lemma 4

Suppose $\mathcal{D} \longrightarrow \mathcal{M}$ is a wide étale subcategory. Let $U, W \in \mathcal{A}$. Then $U \in \mathcal{D}W$ iff $\mathcal{D}U \subseteq \mathcal{D}W$.

Proof

Suppose $U \in \mathcal{D}W$, so that $U = \frac{VW}{S}$, V is in $\mathcal{D}$. Let $\frac{XU}{R}$ be an arbitrary element of $\mathcal{D}U$. By Lemma 2 the domain restriction is in $\mathcal{D}$ (diagram). The other two top horizontals are also in $\mathcal{D}$.

$$\begin{array}{ccccccc}
 & & & U & & & \\
 W & \xrightarrow{\quad} & U & \xrightarrow{\quad} & R & \xrightarrow{X} & Q \\
 & \searrow U & \downarrow U & \searrow U & & & \\
 & & S & \xrightarrow{V} & T & &
 \end{array}$$

Thus, the composite $\frac{XU}{R}$ is in $\mathcal{D}$. By Lemma 3 $\frac{XU}{R} \in \mathcal{D}W$.

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Presheaf of cosets for wide étale $\mathcal{D} \longrightarrow \mathcal{M}$

$$\hat{I}/\mathcal{D}(R) = \{ \mathcal{D}U \mid \text{Ann}(R) = \text{Ann}(U) \}$$

Transition along $W : Q \longrightarrow R$ maps $\mathcal{D}U \mapsto \mathcal{D}\frac{UW}{R}$.

Well-defined: suppose $\mathcal{D}U_1 = \mathcal{D}U_2$. Then $U_1 \in \mathcal{D}U_2$, so $\frac{U_1W}{R} \in \mathcal{D}\frac{U_2W}{R}$. By Lemma 4 we have $\mathcal{D}\frac{U_1W}{R} \subseteq \mathcal{D}\frac{U_2W}{R}$.

The other inclusion is just the same.

The quotient $q : \hat{I} \longrightarrow \hat{I}/\mathcal{D}$

We have a map

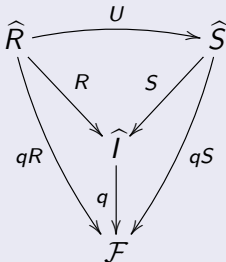
$$q_R : \hat{I}(R) \longrightarrow \hat{I}/\mathcal{D}(R); U \mapsto \mathcal{D}U.$$

Naturality is immediate, so we have a legitimate quotient q in $\mathcal{M}$. The corresponding element $q \in \hat{I}/\mathcal{D}(I)$ is the coset $\mathcal{D}I$.

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Principal fiber of $q : \hat{I} \longrightarrow \mathcal{F}$, where $\mathcal{F}$ is a presheaf

A morphism $U : R \longrightarrow S$ is in the principal fiber of q just when the outside triangle commutes (the inside one commutes iff $R = \frac{SU}{S} = U$).

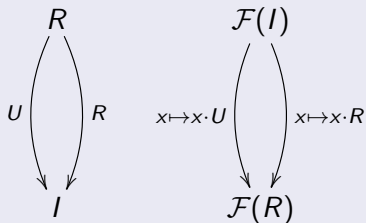


Let $PF(q)$ denote the principal fiber of q .

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Equivalently, let $q \in \mathcal{F}(I)$

$U : R \longrightarrow S$ is in $PF(q)$ just when $q \cdot U = q \cdot R$



Proposition

$FP(q)$ is a wide étale subcategory of $\mathcal{M}$.

Polar decomposition 12

Lemma 5

Let $q \in \mathcal{F}(I)$. For any $U, W \in \widehat{I}(R)$, we have $U \in PF(q)W$ iff $q \cdot U = q \cdot W$.

Proof

Suppose $U \in PF(q)W$. Then we have a diagram like this, where $V \in PF(q)$, so $q \cdot V = q \cdot S$.

$$\begin{array}{ccccc} & & U & & \\ & \curvearrowright & & \curvearrowleft & \\ R & \xrightarrow{W} & S & \xrightarrow{V} & I \\ & & S & & \end{array}$$

Then

$$q \cdot U = (q \cdot V) \cdot W = (q \cdot S) \cdot W = q \cdot \left(\frac{SW}{S} \right) = q \cdot W.$$

Proof continued...

Suppose $q \cdot U = q \cdot W$. Then we have a diagram like this:

$$\begin{array}{ccccc} & & U & & \\ & \nearrow & & \searrow & \\ R & \xrightarrow{W} & W & \xrightarrow{U} & I \\ & \nwarrow & & \nearrow & \\ & & W & & \end{array}$$

Then $U \in PF(q)$, whence $U = UW \in PF(q)W$ (Lemma 3 again).

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The comparison map $\varepsilon(q) = \varepsilon$ associated with $q \in \mathcal{F}(I)$

Let $\varepsilon_R(FP(q)U) = q_R(U) = q \cdot U$.

By Lemma 5 ε_R is a well-defined injective map.

$$\begin{array}{ccccc}
 & \widehat{I}(R) & & \widehat{I} & \\
 U \mapsto PF(q)U \swarrow & & q_R \searrow & \swarrow & q \searrow \\
 \widehat{I}/PF(q)(R) & \xrightarrow{\varepsilon_R} & \mathcal{F}(R) & \widehat{I}/PF(q) & \xrightarrow{\varepsilon} & \mathcal{F}
 \end{array}$$

ε is an isomorphism iff q is an epimorphism.

Proof of Theorem

It remains to show that a given wide étale subcategory $\mathcal{D}$ is recovered as the principal fiber of the quotient $q : \hat{I} \rightarrow \hat{I}/\mathcal{D}$.

A morphism $U : R \rightarrowtail S$ of $\mathcal{M}$ is in $PF(q)$ iff

$\mathcal{D}U = \mathcal{D}I \cdot U = \mathcal{D}I \cdot R = \mathcal{D}R$, which is so iff $U \in \mathcal{D}R$ and

$R \in \mathcal{D}U$. If $U : R \rightarrowtail S$ is in $\mathcal{D}$, then the isomorphism

$U : R \rightarrowtail U$ and its inverse $R : U \rightarrowtail R$ are in $\mathcal{D}$. It follows that $U \in \mathcal{D}R$ and $R \in \mathcal{D}U$, so $\mathcal{D}$ is a subcategory of $PF(q)$.

Conversely, suppose $U : R \rightarrowtail S$ is a morphism of $\mathcal{M}$ satisfying $U \in \mathcal{D}R$. Then we have a diagram, where $V \in \mathcal{D}$.

$$\begin{array}{ccccc} & & U & & \\ & \curvearrowright & & \curvearrowleft & \\ R & \xrightarrow{R} & T & \xrightarrow{V} & I \end{array}$$

Therefore, $U : R \rightarrowtail I$ is in $\mathcal{D}$.

By the étale property $U : R \rightarrowtail S$ is in $\mathcal{D}$.

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The presheaf of positive operators $\hat{I}^+ : \mathcal{M}^{\text{op}} \longrightarrow \text{Set}$

$0 \leq A$ means A is self-adjoint, non-negative spectrum. Let

$$\hat{I}^+(R) = \{ 0 \leq A \mid \text{Ann}(A) = \text{Ann}(R) \}$$

Transition in $\hat{I}^+$ along $U : R \longrightarrow S$ of $\mathcal{M}$: let $A \in \hat{I}^+(S)$.
Therefore, $\sqrt{A} \in \hat{I}^+(S)$.

Define $A \cdot U = \left(\frac{\sqrt{A}U}{S}\right)^* \frac{\sqrt{A}U}{S} = \frac{U^*AU}{S^*S}$, which is positive. If
 $U = SX$, then $\frac{\sqrt{A}U}{S} = \sqrt{A}X$ so $A \cdot U = (\sqrt{A}X)^* \sqrt{A}X = X^*AX$.

$$\begin{array}{ccc} R & \xrightarrow{U} & S \\ & \searrow \frac{\sqrt{A}U}{S} & \downarrow \sqrt{A} \\ & & I \end{array}$$

Polar decomposition 17

The positive quotient $d : \widehat{I} \longrightarrow \widehat{I}^+$

$$d_R : \widehat{I}(R) \longrightarrow \widehat{I}^+(R) ; d_R(U) = U^*U$$

d is a natural transformation (exercise).

d is an epimorphism: if $\text{Ann}(A) = \text{Ann}(R)$, then $d_R(\sqrt{A}) = A$.

Exercise

By Yoneda, d corresponds to an element of

$$\widehat{I}^+(I) = \{ 0 \leq A \mid \text{Ann}(A) = 0 \}.$$

This element is the unit I .

Polar decomposition 18

The principal fiber of the positive quotient d

A morphism $U : R \twoheadrightarrow S$ of $\mathcal{M}$ is in $PF(d)$ just when

$$U^*U = I \cdot U = I \cdot R = R^*R.$$

This condition is stronger than the annihilator condition: if $U^*U = R^*R$, then

$$\text{Ann}(U) = \text{Ann}(U^*U) = \text{Ann}(R^*R) = \text{Ann}(R).$$

Projections and partial isometries again

Let $\mathcal{P}(\mathcal{A}) = \mathcal{P}$ denote the full subcategory of $\mathcal{M}$ on the projections of $\mathcal{A}$. In other words, the objects of $\mathcal{P}$ are the projections of $\mathcal{A}$, and a morphism $U : P \longrightarrow Q$ is an element $U \in \mathcal{A}$ such that $\text{Ann}(P) = \text{Ann}(U)$ and $U = QU$.

The étale subcategory $\partial \longrightarrow \mathcal{M}$ factors through $\mathcal{P}$ as a wide étale subcategory (next diagram).

Proposition

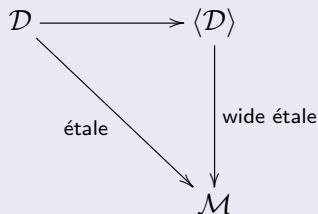
The following is a pullback of subcategories of $\mathcal{M}$.

$$\begin{array}{ccc}
 \partial & \xrightarrow{\text{full}} & PF(d) \\
 \downarrow \text{wide étale} & \searrow \text{étale} & \downarrow \text{wide étale} \\
 \mathcal{P} & \xrightarrow[\text{full}]{\psi(P)=P} & \mathcal{M}
 \end{array}$$

If $U : P \longrightarrow S$ is a morphism of $PF(d)$, where P is a projection, then $U^*U = P^*P = P$, whence U^*U is a projection, so that U is a partial isometry. In particular, the top horizontal is full.

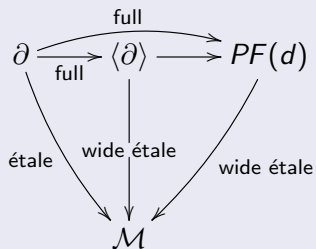
Generated wide étale subcategory

An étale subcategory $\mathcal{D}$ is contained in a smallest wide étale subcategory $\langle \mathcal{D} \rangle$.



If an étale subcategory $\mathcal{D}$ is fully contained in a wide étale subcategory, then $\mathcal{D} \longrightarrow \langle \mathcal{D} \rangle$ is full.

Example



Definition

We shall say that a unital C^* -algebra $\mathcal{A}$ *admits polar decomposition* if $\langle \partial \rangle = PF(d)$ (iff $\langle \partial \rangle \longrightarrow PF(d)$ is full). Equivalently, $\mathcal{A}$ admits polar decomposition if the canonical natural transformation depicted below is an isomorphism (iff it is a monomorphism).

$$\begin{array}{ccc} & \hat{I} & \\ \swarrow & & \searrow d \\ \hat{I}/\langle \partial \rangle & \longrightarrow & \hat{I}^+ \end{array}$$

Comparing quotients across a geometric morphism 1

Presheaves

Essential geometric morphism:

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\psi} & \mathcal{C} \\ \text{Yoneda} \downarrow & & \downarrow \text{Yoneda} \\ \mathcal{D} & \begin{array}{c} \xrightarrow{\psi_!} \\ \xleftarrow{\psi^*} \end{array} & \mathcal{C} \end{array}$$

Let $\mathcal{F}$ be a presheaf on $\mathcal{C}$. What is $\psi_! \psi^*(\mathcal{F})$?

$$\psi_! \psi^*(\mathcal{F})(C) = \{ C \xrightarrow{m} \psi D \xrightarrow{x} \mathcal{F} \} / \sim$$

Comparing quotients across a geometric morphism 2

Example

Essential subtopos.

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\psi(P)=P} & \mathcal{M} \\ \text{Yoneda} \downarrow & & \downarrow \text{Yoneda} \\ \mathcal{P} & \begin{array}{c} \xrightarrow{\psi_!} \\ \xleftarrow{\psi^*} \end{array} & \mathcal{M} \end{array}$$

$\mathcal{F}$ a presheaf on $\mathcal{M}$.

$$\psi_! \psi^*(\mathcal{F})(S) = \{ S \xrightarrow{U} P \xrightarrow{x} \mathcal{F} \} / \sim$$

where P is a projection, $U = PU$, and $x \in \mathcal{F}(P)$.

The spectrum of a commutative C^* -algebra 1

Let $\mathcal{A}$ denote a unital commutative C^* -algebra $\mathcal{A}$.

Definition

Let $\mathcal{O}(\text{Spec}(\mathcal{A}))$ denote the poset of norm-closed ideals of $\mathcal{A}$.

Proposition

$\mathcal{O}(\text{Spec}(\mathcal{A}))$ is a frame. Its points may be identified with the maximal ideals of $\mathcal{A}$.

For any ideal $\mathcal{I} \subseteq \mathcal{A}$ we have

$$\overline{\mathcal{I}} = \bigcap_{\mathcal{I} \subseteq M} M \quad (\text{norm-closure}).$$

The spectrum of a commutative C^* -algebra 2

The functor H

We define a functor H

$$\begin{array}{ccc} \text{DIV}(\mathcal{A}) & & \\ R \mapsto R\mathcal{A} \downarrow & \searrow H & \\ \mathcal{O}(\mathcal{A}) & \xrightarrow{H^*} & \mathcal{O}(\text{Spec}(\mathcal{A})) \end{array} \quad (5)$$

such that

$$H(R) = \overline{R\mathcal{A}} = \bigcap_{R \in M} M.$$

$\mathcal{O}(\mathcal{A})$ denotes the frame of $\mathcal{A}$ -subsets of $\mathcal{A}$ (a subset of $\mathcal{A}$ that is closed under multiplication by elements of $\mathcal{A}$.)

If $U \preceq R$, then $H(U) \subseteq H(R)$, so that H is indeed a functor.

$H(0) = (0)$, $H(I) = \mathcal{A}$, and $H(RS) = H(R) \cap H(S)$.

The spectrum of a commutative C^* -algebra 3

The frame map H^*

The suprema extension H^* is given by the norm-closure of the ideal generated by an $\mathcal{A}$ -subset $\mathcal{X} \subseteq \mathcal{A}$:

$$H^*(\mathcal{X}) = \overline{(\mathcal{X})} = \bigcap_{\mathcal{X} \subseteq M} M.$$

H is flat in the sense that H^* preserves finite infima.

Thus, we have geometric morphisms as follows.

$$\begin{array}{ccc} Sh(\mathrm{Spec}(\mathcal{A})) & \xrightarrow{H^* \dashv H_*} & \mathcal{M}/\widehat{I} \\ \downarrow & & \downarrow \text{étale surjection} \\ \mathrm{SPEC}(\mathcal{A}) & \xrightarrow{\quad} & \mathcal{M} \end{array}$$

GNS representation theory 1

Functionals

A functional on $\mathcal{A}$ is a bounded linear map $\tau : \mathcal{A} \longrightarrow \mathbb{C}$ (complex numbers).

Let $B(\mathcal{A})$ denote the vector space of functionals on $\mathcal{A}$.

$B(\mathcal{A})$ is a Banach space: $\|\tau\| = \sup_{X \neq 0} \frac{|\tau(X)|}{\|X\|}$

$B(\mathcal{A})$ is an $\mathcal{A}$ -module: $\tau U(X) = \tau(XU^*)$

$\|\tau U\| \leq \|\tau\| \|U\|$ holds (exercise).

$B(\mathcal{A})$ is a Banach $\mathcal{A}$ -module.

Positive functional

A functional τ is positive if $0 \leq \tau(A)$ for every positive A .

GNS representation theory 2

Example: sesquilinear form on $\mathcal{A}$

If τ is a functional on $\mathcal{A}$, then $\langle U, V \rangle = \tau(UV^*)$ is a sesquilinear form on $\mathcal{A}$ satisfying

$$\langle UX, V \rangle = \langle U, VX^* \rangle.$$

If τ is positive, then $\langle U, V \rangle$ is a semi inner-product, so that

$\|U\|_2 = \sqrt{\langle U, U \rangle} = \sqrt{\tau(UU^*)}$ is a seminorm on $\mathcal{A}$.

$\|U\|_2$ is not to be confused with the C^* -algebra norm $\|U\|$ of $\mathcal{A}$.

We have $\|U\|_2 = \|U\|$ just when $\tau(UU^*) = \|UU^*\|$.

Definition: sesquilinear form on an $\mathcal{A}$ -module

Let $\mathcal{V}$ be an $\mathcal{A}$ -module. A sesquilinear form on $\mathcal{V}$ must satisfy

$$\langle uX, v \rangle = \langle u, vX^* \rangle.$$

GNS representation theory 3

The Cauchy-Bunyakovsky-Schwartz Lemma (CBS)

If a sesquilinear form on $\mathcal{V}$ is positive and symmetric, then

$$|\langle u, v \rangle|^2 \leq \langle u, u \rangle \langle v, v \rangle .$$

In particular, if τ is positive (whence τ is Hermitian), then

$$|\tau(UV^*)|^2 \leq \tau(UU^*)\tau(VV^*) .$$

Annihilators

$$\text{Ann}(\tau) = \{ V \mid \tau V = 0 \} ,$$

where $\tau V = 0$ means $\tau(UV^*) = 0$ for every $U \in \mathcal{A}$.

Lemma

For any positive τ and V we have $\tau V = 0$ iff $\tau(VV^*) = 0$.

GNS representation theory 4

Proposition

Let τ be a positive functional on $\mathcal{A}$.

Then $U \equiv R \pmod{\text{Ann}(\tau)}$ iff $\tau(UU^*) = \tau(RR^*) = \tau(RU^*)$.

Proof

If $U \equiv R \pmod{\text{Ann}(\tau)}$, then $\tau U = \tau R$, whence $\tau(UU^*) = \tau(RR^*) = \tau(RU^*)$.

Conversely, if these three are equal, then

$\tau(UU^*) + \tau(RR^*) = \tau(RU^*) + \overline{\tau(RU^*)} = \tau(RU^*) + \tau(UR^*)$,
so that $\tau((U - R)(U^* - R^*)) = 0$, whence $U \equiv R \pmod{\text{Ann}(\tau)}$.

The wide étale subcategory of a positive τ

Let τ be a positive functional on $\mathcal{A}$. Then its wide étale subcategory consists of morphisms $U : R \longrightarrow S$ of $\mathcal{M}$ such that $\tau(UU^*) = \tau(RR^*) = \tau(RU^*)$.

Module to presheaf

Let $\widehat{B(\mathcal{A})}(R)$ denote

$$\mathcal{A}\text{-MOD}(R\mathcal{A}, B(\mathcal{A})) \cong \{ \tau \in B(\mathcal{A}) \mid \text{Ann}(R) \subseteq \text{Ann}(\tau) \}.$$

Transition in $\widehat{B(\mathcal{A})}$ along $U : R \twoheadrightarrow S$ is given by

$$\tau \cdot U = \tau \frac{U}{S} = \tau Y; \quad U = SY.$$

$$\tau \cdot U(X) = \tau Y(X) = \tau(XY^*); \quad \begin{array}{ccc} \widehat{R} & \xrightarrow{U} & \widehat{S} \\ \searrow \tau \cdot U & & \swarrow \tau \\ & \widehat{B(\mathcal{A})} & \end{array}$$

$B(\mathcal{A})$ classifies functionals

Let $\mathcal{V}$ be a seminormed $\mathcal{A}$ -module.

$$\mathcal{A}\text{-BDD}(\mathcal{V}, B(\mathcal{A})) \begin{matrix} \xrightarrow{\gamma \mapsto \bar{\gamma}} \\ \xleftarrow{\widehat{\delta} \mapsto \delta} \end{matrix} B(\mathcal{V})$$

$$\bar{\gamma}(v) = \overline{\gamma(v)(I)} ; \widehat{\delta}(v)(U) = \overline{\delta(vU^*)}$$

For instance,

$$\bar{\gamma}(vz) = \overline{\gamma(vz)(I)} = \overline{\gamma(v)(\bar{z})} = \overline{\gamma(v)(I)\bar{z}} = \bar{\gamma}(v)z .$$

GNS representation theory 7

$\widehat{B(\mathcal{A})}$ again

Let $\mathcal{V} = R\mathcal{A}$

$$\begin{array}{ccc} \{\tau \in B(\mathcal{A}) \mid \text{Ann}(R) \subseteq \text{Ann}(\tau)\} & \xrightarrow{\tau \mapsto \bar{\tau}} & B(R\mathcal{A}) \\ & \xleftarrow{\widehat{\delta} \mapsto \delta} & \end{array}$$

$$\bar{\tau}(RX) = \overline{\tau(X^*)} = \tau^\dagger(X) ; \quad \widehat{\delta}(U) = \overline{\delta(RU^*)}$$

$\bar{\tau}$ is well-defined: if $RX = RY$, then $\tau X = \tau Y$ so that $\tau(X^*) = \tau X(I) = \tau Y(I) = \tau(Y^*)$.

Both sides are $\mathcal{A}$ -modules (actually $\mathcal{A}$ -algebras): the right side in the usual way (1), and the left side by $\tau \bullet U(X) = \tau(UX)$. Note that $\text{Ann}(\tau) \subseteq \text{Ann}(\tau \bullet U)$. Furthermore, the bijection respects the two actions.

On our way...

We are prepared to analyze the geometric morphism

$$\begin{array}{c} \mathcal{M} / \widehat{B(\mathcal{A})} \\ \downarrow \\ \mathcal{M} \end{array}$$

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Thank you