More toposes and C^*-algebras

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Introduction 1

A unital C^*-algebra $\mathcal{A}$ has associated with it a category $\mathcal{M}(\mathcal{A}) = \mathcal{M}$, which I call the division category of (the underlying ring of) $\mathcal{A}$. A typical object of $\mathcal{M}$ is an element of $\mathcal{A}$, denoted R, S etc., and a morphism $U : R \longrightarrow S$ is an element U of $\mathcal{A}$ such that $\mathrm{Ann}(R) = \mathrm{Ann}(U)$ and $U = SX$ for some X, where $\mathrm{Ann}(R) = \{X \in \mathcal{A} \mid RX = 0\}$. (There is also a category with the same objects but with morphisms U such that $\mathrm{Ann}(R) \subseteq \mathrm{Ann}(U)$; however, I prefer to work with $\mathcal{M}$ as it offers a bit more flexibility.) For instance, for any element R of $\mathcal{A}$, the objects R and R^*R are isomorphic in $\mathcal{M}$. It is sometimes good to know that there is a weak equivalence functor from $\mathcal{M}$ to the category of principal right ideals of $\mathcal{A}$ and injective $\mathcal{A}$-module maps between them mapping an object R of $\mathcal{M}$ to the right ideal $R\mathcal{A}$ it generates.
Let $\mathscr{M}$ denote the topos of presheaves on $\mathcal{M}$.

Introduction 2

We shall discuss three aspects beginning with:
1. *The positive quotient and polar decomposition*. An operator T on Hilbert space has a polar form $T = UA$, where A is positive, U is a partial isometry, and $\mathrm{Ann}(T) = \mathrm{Ann}(U)$. Let us say that a C^*-algebra $\mathcal{A}$ admits polar decomposition if every element of it has such a decompostion. It follows that if $\mathcal{A}$ admits polar decomposition, then necessarily $\mathcal{A}$ is supported as I call it. (Moreover, it follows that $A = |T| = \sqrt{T^*T}$ and that the decomposition is unique.) How do we generalize polar decomposition to all C^*-algebras, and not just the supported ones? This question has an answer in terms of a canonical correspondence between wide étale subcategories of $\mathcal{M}$ and the quotients in $\mathscr{M}$ of the representable presheaf associated with unit of $\mathcal{A}$.
2. *The Gelfand spectrum of a commutative C^*-algebra: 1-dimensional representations*. How this is related to $\mathcal{M}$ follows a pattern similar to the Zariski spectrum of a commutative ring.

3. *GNS-representation theory (in any dimension), and the spectrum of an arbitrary C^*-algebra*. One of our tools is what we shall call *a seminormed $\mathcal{A}$-module*, by which we mean a right $\mathcal{A}$-module $\mathcal{V}$ that carries a seminorm such that

$$\forall v \in \mathcal{V}\,,\ \forall U \in \mathcal{A}\,,\ \|vU\| \leq \|v\|\,\|U\|\,.$$

(Thus, if $\|v\| = 0$, then $\|vU\| = 0$ for every U.) The functional dual $B(\mathcal{A})$, consisting of bounded $\mathbb{C}$-linear maps $\tau : \mathcal{A} \longrightarrow \mathbb{C}$, is an important $\mathcal{A}$-module especially for GNS-theory. Its right action is defined by

$$\tau U(X) = \tau(XU^*)\,;\ \tau \in B(\mathcal{A})\,. \tag{1}$$

What is more this action satisfies $\|\tau U\| \leq \|\tau\|\,\|U\|$, so $B(\mathcal{A})$ is a normed $\mathcal{A}$-module. $B(\mathcal{A})$ classifies functionals in the sense that for any seminormed $\mathcal{A}$-module $\mathcal{V}$ there is a natural isomorphism

$$B(\mathcal{V}) \cong \mathcal{A}\text{-Bdd}(\mathcal{V}, B(\mathcal{A}))\,. \tag{2}$$

Another important tool is how to parameterize a (right) $\mathcal{A}$-module as an object of $\mathscr{M}$: if $\mathcal{V}$ is an $\mathcal{A}$-module, then

$$\widehat{\mathcal{V}}(R) = \mathcal{A}\text{-Mod}(R\mathcal{A}, \mathcal{V}) \cong \{ v \in \mathcal{V} \mid \mathrm{Ann}(R) \subseteq \mathrm{Ann}(v) \}$$

is a presheaf on $\mathcal{M}$. If $\mathcal{V}$ is a seminormed $\mathcal{A}$-module, then an $\mathcal{A}$-module map $R\mathcal{A} \longrightarrow \mathcal{V}$ is necessarily bounded. Therefore, we have

$$\widehat{\mathcal{V}}(R) = \mathcal{A}\text{-Bdd}(R\mathcal{A}, \mathcal{V}) \text{ (bounded } \mathcal{A}\text{-module maps)}\,. \tag{3}$$

The presheaf associated with $B(\mathcal{A})$ is given by

$$\widehat{B(\mathcal{A})}(R) \overbrace{=}^{(3)} \mathcal{A}\text{-Bdd}(R\mathcal{A}, B(\mathcal{A})) \overbrace{\cong}^{(2)} B(R\mathcal{A})\,. \tag{4}$$

From this point of view transition in $\widehat{B(\mathcal{A})}$ along a morphism $U : R \longrightarrow S$ of $\mathcal{M}$ is achieved in the following way by passing to principal right ideals.

$$\begin{tikzcd} R\mathcal{A} \arrow[r, bend left, "RX \mapsto UX"] \arrow[dr, "\tau \cdot U"'] & S\mathcal{A} \arrow[d, "\tau"] \\ & \mathbb{C} \end{tikzcd} \qquad \tau \cdot U(RX) = \tau(UX)$$

The upshot is that $\widehat{B(\mathcal{A})}$ is the presheaf of functional germs on $\mathcal{A}$, suggesting that $\widehat{B(\mathcal{A})}$ may be interpreted as the complex numbers object of the topos $\mathscr{M}$. In any case, $\widehat{B(\mathcal{A})}$ is a ring object with a conjugation operation (such that $\overline{\tau}(SX) = \overline{\tau(SX)}$) internal to $\mathscr{M}$.

Supported C^*-algebra 1

Support/cosupport projection

- A support projection $C(T)$ satisfies $C(T) \leq P$ iff $T = PT$
- A cosupport projection $N(T)$ satisfies $P \leq N(T)$ iff $TP = 0$

Support hypothesis

We shall say that a C^*-algebra $\mathcal{A}$ *is supported* if:

1. every $T \in \mathcal{A}$ has a support projection $C(T)$ such that
2. $\forall S, T, R : \; C(S) \leq C(T) \Rightarrow C(RS) \leq C(RT)$ (stability).

Equivalent cosupport hypothesis

1. every T has a cosupport projection $N(T)$ such that
2. $\forall S, T, R : \; N(S) \leq N(T) \Rightarrow N(SR) \leq N(TR)$.

For any $T \in \mathcal{A}$ let $\mathrm{Ann}(T) = \{ X \in \mathcal{A} \mid TX = 0 \}$

Blackadar [1]

For every $T \in \mathcal{A}$ there is a projection P such that $\mathrm{Ann}(T) = \mathrm{Ann}(P)$. This implies $P = C(T^*)$.

Rickart

For every $T \in \mathcal{A}$ there is an idempotent E such that $\mathrm{Ann}(T) = \mathrm{Ann}(E)$.

Theorem

The three notions supported, Blackadar, and Rickart are equivalent.

Kerzman-Stein formula [2]: $C(E) = E(E + E^* - I)^{-1}$

Supported C^*-algebra 3

von Neumann algebra

$B(\mathcal{H})$ and more generally any von Neumann algebra is supported in this sense.

$C(X)$

What about the C^*-algebra $C(X)$ of continuous functions on a (connected) compact Hausdorff space X?
Generally, $C(X)$ is not supported.

Definition of $\mathcal{M}(\mathcal{A}) = \mathcal{M}$

Objects: elements of $\mathcal{A}$
Morphisms: $U : R \longrightarrow S$ such that $\mathrm{Ann}(R) = \mathrm{Ann}(U)$, and $U \in S\mathcal{A}$ (also written $S \mid U$).

$$
\begin{tikzcd}
R \arrow[r, "U"] \arrow[dr, "\frac{VU}{S}"'] & S \arrow[d, "V"] & R \arrow[r, "R"] \arrow[dr, "\frac{UR}{R}=U"'] & R \arrow[d, "U"] & R \arrow[r, "U"] \arrow[dr, "\frac{SU}{S}=U"'] & S \arrow[d, "S"] \\
& T & & S & & S
\end{tikzcd}
$$

If $U = SX$, then $\frac{VU}{S} = VX$.

The division category 2

The divisor preorder

Let $\mathrm{DIV}(\mathcal{A})$ denote the divisor preorder $\preceq$ on (the underlying ring of) $\mathcal{A}$ such that $U \preceq R$ if $U = RX$ for some X .

Proposition

Then $\mathrm{DIV}(\mathcal{A}) \simeq \mathcal{M}/I$.

Proof

Objects: R maps to $R : R \longrightarrow I$, and in the other direction $U : R \longrightarrow I$ to U . The top morphism is an isomorphism.

$$\begin{tikzcd} R \arrow[r, bend right, "U"'] \arrow[dr, "U"'] & U \arrow[l, bend right, "R"'] \arrow[d, "U"] \\ & I \end{tikzcd}$$

The division category 3

Definition

Let $\mathscr{M}$ denote the topos of presheaves on $\mathcal{M}$.

Representable presheaf

Let $S \in \mathcal{A}$. $\widehat{S} : \mathcal{M}^{\mathrm{op}} \longrightarrow Set$
$\widehat{S}(R) = \mathcal{M}(R, S) = \{ \text{morphisms } U : R \longrightarrow S \text{ of } \mathcal{M} \}$
$= \{ U \mid \mathrm{Ann}(R) = \mathrm{Ann}(U) \text{ and } S \mid U \}$.

Representable presheaf associated with the unit I

$\widehat{I} : \mathcal{M}^{\mathrm{op}} \longrightarrow Set$
$\widehat{I}(R) = \{ U \in \mathcal{A} \mid \mathrm{Ann}(R) = \mathrm{Ann}(U) \}$
Note: $R \in \widehat{I}(R)$ so $\widehat{I}$ has global support.

Proposition

$\mathscr{M}$ is an étendue.

Proof

A right $\mathcal{A}$-subset of $\mathcal{A}$ is the same as a downset of $\mathrm{DIV}(\mathcal{A})$. Let $\mathcal{O}(\mathcal{A})$ denote the frame of right $\mathcal{A}$-subsets of $\mathcal{A}$, with topos $Sh(\mathcal{A})$. $\mathcal{O}(\mathcal{A})$ and the frame $\mathrm{Sub}(\widehat{I})$ are isomorphic.

$$
\begin{tikzcd}
Sh(\mathcal{A}) \arrow[r, bend left] \arrow[d, bend left] & \mathscr{M}/\widehat{I} \arrow[l, bend left] \arrow[d, bend left] \\
\mathrm{SET}^{\mathrm{DIV}(\mathcal{A})^{\mathrm{op}}} \arrow[u, bend left] \arrow[r, bend left] & \mathrm{SET}^{\mathcal{M}/I^{\mathrm{op}}} \arrow[u, bend left] \arrow[l, bend left]
\end{tikzcd}
$$

Polar decomposition 1

Definition

A subcategory $\mathcal{D} \longrightarrow \mathcal{M}$ is *an étale subcategory* if:

1. for any two objects R, S of $\mathcal{D}$, if $S \mid R$, then the morphism $R : R \longrightarrow S$ is in $\mathcal{D}$;
2. in a triangle
$$\begin{tikzcd} R \arrow[r, "U"] \arrow[dr, "\frac{VU}{S}"'] & S \arrow[d, "V"] \\ & T \end{tikzcd}$$
of composable morphisms of $\mathcal{M}$ if $V, \frac{VU}{S} \in \mathcal{D}$, then $U \in \mathcal{D}$.

Example: projections and partial isometries

Let $\partial \longrightarrow \mathcal{M}$ denote the subcategory whose objects are all the projections of $\mathcal{A}$, such that a morphism $U : P \longrightarrow Q$ is in ∂ just when U is a partial isometry.

Polar decomposition 2

Proposition

A morphism $U : P \longrightarrow Q$ of $\mathcal{M}$ is in ∂ iff $P = U^*U$.
$\partial \longrightarrow \mathcal{M}$ is an étale subcategory.

Proof

If U is a partial isometry, then it follows that $P = C(U^*) = U^*U$.
Conversely, if $P = U^*U$, then U is a partial isometry.

Definition

A subcategory $\mathcal{D} \longrightarrow \mathcal{M}$ is *wide* if it every object of $\mathcal{M}$ is in $\mathcal{D}$.

Polar decomposition 3

Theorem

Wide étale subcategories of $\mathcal{M}$ and quotients of $\widehat{I}$ correspond.

$$\begin{tikzcd} \text{Wide étale subcats} \arrow[r, bend left, "\text{quotient of cosets}"] & \text{quotients of } \widehat{I} \arrow[l, bend left, "\text{principal fiber}"] \end{tikzcd}$$

Example

Let $\mathcal{I} \subseteq \mathcal{A}$ be a right ideal. Then the morphisms $U : R \longrightarrow S$ such that $U \equiv R \pmod{\mathcal{I}}$ form a wide étale subcategory of $\mathcal{M}$.

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Lemma 1

Suppose $\mathcal{D} \longrightarrow \mathcal{M}$ is a wide étale subcategory. Suppose we are given S, V such that $\mathrm{Ann}(S) = \mathrm{Ann}(V)$. Then t.f.a.e.

1. the morphism $V : S \longrightarrow I$ is in $\mathcal{D}$;
2. for any T, if $T \mid V$, then $V : S \longrightarrow T$ is in $\mathcal{D}$;
3. the morphism $V : S \longrightarrow V$ is in $\mathcal{D}$.

Lemma 2

A wide étale subcategory of $\mathcal{M}$ is closed under domain restrictions.

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Proof

The domain restriction of a morphism is the following factorization. Let $U = \frac{VR}{S}$.

$$\begin{tikzcd} R \arrow[r, "U"] \arrow[d, "R"'] & U \arrow[d, "U"] \\ S \arrow[r, "V"] & T \end{tikzcd}$$

If the morphism V is in $\mathcal{D}$, then $R \xrightarrow{U} U$ is in $\mathcal{D}$.

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Coset of a subcategory $\mathcal{D} \longrightarrow \mathcal{M}$

Let $U \in \mathcal{A}$. Let

$$\mathcal{D}U = \{ \frac{VU}{S} \mid S \mid U \text{ and } V : S \longrightarrow T \in \mathcal{D} \}$$

We call $\mathcal{D}U$ a coset of $\mathcal{D}$. $\mathcal{D}U$ is a subset of $\mathcal{A}$.

Example

$$\mathcal{D}I = \{ \frac{V}{S} \mid S \mid I \text{ and } V : S \longrightarrow T \in \mathcal{D} \}$$

Lemma 3

Suppose $\mathcal{D} \longrightarrow \mathcal{M}$ is a subcategory.

1. If a morphism $V : S \longrightarrow T$ of $\mathcal{M}$ is in $\mathcal{D}$, then $V \in \mathcal{D}S$.
2. $U \in \mathcal{D}U$ iff there is an object S of $\mathcal{D}$ such that $S \mid U$.

Proof

1. Use $S \xrightarrow{S} S \xrightarrow{V} T$ equals V.
2. Use $U \xrightarrow{U} S \xrightarrow{S} S$ equals U.

Lemma 4

Suppose $\mathcal{D} \longrightarrow \mathcal{M}$ is a wide étale subcategory. Let $U, W \in \mathcal{A}$. Then $U \in \mathcal{D}W$ iff $\mathcal{D}U \subseteq \mathcal{D}W$.

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Proof

Suppose $U \in \mathcal{D}W$, so that $U = \frac{VW}{S}$, V is in $\mathcal{D}$. Let $\frac{XU}{R}$ be an arbitrary element of $\mathcal{D}U$. By Lemma 2 the domain restriction is in $\mathcal{D}$ (diagram). The other two top horizontals are also in $\mathcal{D}$.

$$\begin{tikzcd}
W \arrow[r, "U"'] \arrow[rr, bend left, "U"] \arrow[d, "W"'] & U \arrow[r, "U"'] \arrow[d, "U"] & R \arrow[r, "X"] & Q \\
S \arrow[r, "V"] & T & &
\end{tikzcd}$$

Thus, the composite $\frac{XU}{R}$ is in $\mathcal{D}$. By Lemma 3 $\frac{XU}{R} \in \mathcal{D}W$.

Presheaf of cosets for wide étale $\mathcal{D} \longrightarrow \mathcal{M}$

$$\widehat{I}/\mathcal{D}\,(R) = \{\, \mathcal{D}U \mid \mathrm{Ann}(R) = \mathrm{Ann}(U) \,\}$$

Transition along $W : Q \longrightarrow R$ maps $\mathcal{D}U \mapsto \mathcal{D}\frac{UW}{R}$.
Well-defined: suppose $\mathcal{D}U_1 = \mathcal{D}U_2$. Then $U_1 \in \mathcal{D}U_2$, so $\frac{U_1 W}{R} \in \mathcal{D}\frac{U_2 W}{R}$. By Lemma 4 we have $\mathcal{D}\frac{U_1 W}{R} \subseteq \mathcal{D}\frac{U_2 W}{R}$.
The other inclusion is just the same.

The quotient $q : \widehat{I} \longrightarrow \widehat{I}/\mathcal{D}$

We have a map

$$q_R : \widehat{I}\,(R) \longrightarrow \widehat{I}/\mathcal{D}\,(R)\,;\;\; U \mapsto \mathcal{D}U\,.$$

Naturality is immediate, so we have a legitimate quotient q in $\mathscr{M}$.
The corresponding element $q \in \widehat{I}/\mathcal{D}(I)$ is the coset $\mathcal{D}I$.

Principal fiber of $q : \widehat{I} \longrightarrow \mathcal{F}$, where $\mathcal{F}$ is a presheaf

A morphism $U : R \longrightarrow S$ is in the principal fiber of q just when the outside triangle commutes (the inside one commutes iff $R = \frac{SU}{S} = U$).

$$\begin{tikzcd}
\widehat{R} \arrow[rr, bend left, "U"] \arrow[dr, "R"] \arrow[ddr, bend right, "qR"'] & & \widehat{S} \arrow[dl, "S"'] \arrow[ddl, bend left, "qS"] \\
& \widehat{I} \arrow[d, "q"] & \\
& \mathcal{F} &
\end{tikzcd}$$

Let $PF(q)$ denote the principal fiber of q .

Equivalently, let $q \in \mathcal{F}(I)$

$U : R \longrightarrow S$ is in $PF(q)$ just when $q \cdot U = q \cdot R$

$$\begin{tikzcd} R \arrow[d, bend right, "U"'] \arrow[d, bend left, "R"] \\ I \end{tikzcd} \qquad \begin{tikzcd} \mathcal{F}(I) \arrow[d, bend right, "x \mapsto x \cdot U"'] \arrow[d, bend left, "x \mapsto x \cdot R"] \\ \mathcal{F}(R) \end{tikzcd}$$

Proposition

$FP(q)$ is a wide étale subcategory of $\mathcal{M}$.

Lemma 5

Let $q \in \mathcal{F}(I)$. For any $U, W \in \widehat{I}(R)$, we have $U \in PF(q)W$ iff $q \cdot U = q \cdot W$.

Proof

Suppose $U \in PF(q)W$. Then we have a diagram like this, where $V \in PF(q)$, so $q \cdot V = q \cdot S$.

$$
\begin{tikzcd}
R \arrow[r, "W"'] \arrow[rr, bend left, "U"] & S \arrow[r, shift left, "V"] \arrow[r, shift right, "S"'] & I
\end{tikzcd}
$$

Then

$$q \cdot U = (q \cdot V) \cdot W = (q \cdot S) \cdot W = q \cdot (\frac{SW}{S}) = q \cdot W \, .$$

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Proof continued...

Suppose $q \cdot U = q \cdot W$. Then we have a diagram like this:

$$\begin{tikzcd} R \arrow[r, "W"'] \arrow[rr, bend left, "U"] & W \arrow[r, shift left, "U"] \arrow[r, shift right, "W"'] & I \end{tikzcd}$$

Then $U \in PF(q)$, whence $U = UW \in PF(q)W$ (Lemma 3 again).

The comparison map $\varepsilon(q) = \varepsilon$ associated with $q \in \mathcal{F}(I)$

Let $\varepsilon_R(FP(q)U) = q_R(U) = q \cdot U$.
By Lemma 5 ε_R is a well-defined injective map.

$$
\begin{tikzcd}
 & \widehat{I}(R) \arrow[dl, "U \mapsto PF(q)U"'] \arrow[dr, "q_R"] & \\
\widehat{I}/PF(q)(R) \arrow[rr, tail, "\varepsilon_R"] & & \mathcal{F}(R)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \widehat{I} \arrow[dl] \arrow[dr, "q"] & \\
\widehat{I}/PF(q) \arrow[rr, tail, "\varepsilon"] & & \mathcal{F}
\end{tikzcd}
$$

ε is an isomorphism iff q is an epimorphism.

Proof of Theorem

It remains to show that a given wide étale subcategory $\mathcal{D}$ is recovered as the principal fiber of the quotient $q : \widehat{I} \longrightarrow \widehat{I}/\mathcal{D}$.
A morphism $U : R \longrightarrow S$ of $\mathcal{M}$ is in $PF(q)$ iff
$\mathcal{D}U = \mathcal{D}I \cdot U = \mathcal{D}I \cdot R = \mathcal{D}R$, which is so iff $U \in \mathcal{D}R$ and $R \in \mathcal{D}U$. If $U : R \longrightarrow S$ is in $\mathcal{D}$, then the isomorphism $U : R \longrightarrow U$ and its inverse $R : U \longrightarrow R$ are in $\mathcal{D}$. It follows that $U \in \mathcal{D}R$ and $R \in \mathcal{D}U$, so $\mathcal{D}$ is a subcategory of $PF(q)$.
Conversely, suppose $U : R \longrightarrow S$ is a morphism of $\mathcal{M}$ satisfying $U \in \mathcal{D}R$. Then we have a diagram, where $V \in \mathcal{D}$.

$$\begin{tikzcd} R \arrow[r, "R"'] \arrow[rr, bend left, "U"] & T \arrow[r, "V"'] & I \end{tikzcd}$$

Therefore, $U : R \longrightarrow I$ is in $\mathcal{D}$.
By the étale property $U : R \longrightarrow S$ is in $\mathcal{D}$.

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The presheaf of positive operators $\widehat{I}^{+} : \mathcal{M}^{\mathrm{op}} \longrightarrow Set$

$0 \leq A$ means A is self-adjoint, non-negative spectrum. Let

$$\widehat{I}^{+}(R) = \{\, 0 \leq A \mid \mathrm{Ann}(A) = \mathrm{Ann}(R) \,\}$$

Transition in $\widehat{I}^{+}$ along $U : R \longrightarrow S$ of $\mathcal{M}$: let $A \in \widehat{I}^{+}(S)$.
Therefore, $\sqrt{A} \in \widehat{I}^{+}(S)$.
Define $A \cdot U = (\frac{\sqrt{A}U}{S})^{*}\frac{\sqrt{A}U}{S} = \frac{U^{*}AU}{S^{*}S}$, which is positive. If
$U = SX$, then $\frac{\sqrt{A}U}{S} = \sqrt{A}X$ so $A \cdot U = (\sqrt{A}X)^{*}\sqrt{A}X = X^{*}AX$.

$$\begin{tikzcd}
R \arrow[r, "U"] \arrow[dr, "\frac{\sqrt{A}U}{S}"'] & S \arrow[d, "\sqrt{A}"] \\
 & I
\end{tikzcd}$$

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The positive quotient $d : \widehat{I} \longrightarrow \widehat{I}^{+}$

$d_R : \widehat{I}(R) \longrightarrow \widehat{I}^{+}(R)$; $d_R(U) = U^* U$
d is a natural transformation (exercise).
d is an epimorphism: if $\mathrm{Ann}(A) = \mathrm{Ann}(R)$, then $d_R(\sqrt{A}) = A$.

Exercise

By Yoneda, d corresponds to an element of

$$\widehat{I}^{+}(I) = \{\, 0 \leq A \mid \mathrm{Ann}(A) = 0 \,\}.$$

This element is the unit I.

The principal fiber of the positive quotient d

A morphism $U : R \longrightarrow S$ of $\mathcal{M}$ is in $PF(d)$ just when

$$U^* U = I \cdot U = I \cdot R = R^* R \,.$$

This condition is stronger than the annihilator condition: if $U^* U = R^* R$, then

$$\mathrm{Ann}(U) = \mathrm{Ann}(U^* U) = \mathrm{Ann}(R^* R) = \mathrm{Ann}(R) \,.$$

Projections and partial isometries again

Let $\mathcal{P}(\mathcal{A}) = \mathcal{P}$ denote the full subcategory of $\mathcal{M}$ on the projections of $\mathcal{A}$. In other words, the objects of $\mathcal{P}$ are the projections of $\mathcal{A}$, and a morphism $U : P \longrightarrow Q$ is an element $U \in \mathcal{A}$ such that $\mathrm{Ann}(P) = \mathrm{Ann}(U)$ and $U = QU$.
The étale subcategory $\partial \longrightarrow \mathcal{M}$ factors through $\mathcal{P}$ as a wide étale subcategory (next diagram).

Proposition

The following is a pullback of subcategories of $\mathcal{M}$.

$$\begin{tikzcd}
\partial \arrow[r, "\text{full}"] \arrow[d, "\text{wide étale}"'] \arrow[dr, "\text{étale}"] & PF(d) \arrow[d, "\text{wide étale}"] \\
\mathcal{P} \arrow[r, "\psi(P)=P", "\text{full}"'] & \mathcal{M}
\end{tikzcd}$$

If $U : P \longrightarrow S$ is a morphism of $PF(d)$, where P is a projection, then $U^*U = P^*P = P$, whence U^*U is a projection, so that U is a partial isometry. In particular, the top horizontal is full.

Generated wide étale subcategory

An étale subcategory $\mathcal{D}$ is contained in a smallest wide étale subcategory $\langle\mathcal{D}\rangle$.

$$\begin{tikzcd} \mathcal{D} \arrow[r] \arrow[dr, "\text{étale}"'] & \langle\mathcal{D}\rangle \arrow[d, "\text{wide étale}"] \\ & \mathcal{M} \end{tikzcd}$$

If an étale subcategory $\mathcal{D}$ is fully contained in a wide étale subcategory, then $\mathcal{D} \longrightarrow \langle\mathcal{D}\rangle$ is full.

Example

$$
\begin{tikzcd}
\partial \arrow[r, "\text{full}"'] \arrow[rr, bend left, "\text{full}"] \arrow[dr, bend right, "\text{étale}"'] & \langle\partial\rangle \arrow[r] \arrow[d, "\text{wide étale}"] & PF(d) \arrow[dl, bend left, "\text{wide étale}"] \\
& \mathcal{M} &
\end{tikzcd}
$$

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Definition

We shall say that a unital C^*-algebra $\mathcal{A}$ *admits polar decomposition* if $\langle \partial \rangle = PF(d)$ (iff $\langle \partial \rangle \longrightarrow PF(d)$ is full). Equivalently, $\mathcal{A}$ admits polar decomposition if the canonical natural transformation depicted below is an isomorphism (iff it is a monomorphism).

$$
\begin{tikzcd}
 & \widehat{I} \arrow[dl] \arrow[dr, two heads, "d"] & \\
\widehat{I}/\langle \partial \rangle \arrow[rr, two heads] & & \widehat{I}^{+}
\end{tikzcd}
$$

Presheaves

Essential geometric morphism:

$$\begin{tikzcd} \mathcal{D} \arrow[r, "\psi"] \arrow[d, "\text{Yoneda}"'] & \mathcal{C} \arrow[d, "\text{Yoneda}"] \\ \mathscr{D} \arrow[r, bend left, "\psi_!"] & \mathscr{C} \arrow[l, bend left, "\psi^*"] \end{tikzcd}$$

Let $\mathcal{F}$ be a presheaf on $\mathcal{C}$. What is $\psi_!\psi^*(\mathcal{F})$?

$$\psi_!\psi^*(\mathcal{F})(C) = \{\, C \xrightarrow{m} \psi D \xrightarrow{x} \mathcal{F} \,\}/\sim$$

Example

Essential subtopos.

$$\begin{tikzcd} \mathcal{P} \arrow[r, "\psi(P)=P"] \arrow[d, "\text{Yoneda}"'] & \mathcal{M} \arrow[d, "\text{Yoneda}"] \\ \mathscr{P} \arrow[r, bend left, "\psi_!"] & \mathscr{M} \arrow[l, bend left, "\psi^*"] \end{tikzcd}$$

$\mathcal{F}$ a presheaf on $\mathcal{M}$.

$$\psi_!\psi^*(\mathcal{F})(S) = \{\, S \xrightarrow{U} P \xrightarrow{x} \mathcal{F} \,\}/\sim$$

where P is a projection, $U = PU$, and $x \in \mathcal{F}(P)$.

The spectrum of a commutative C^*-algebra 1

Let $\mathcal{A}$ denote a unital commutative C^*-algebra $\mathcal{A}$.

Definition

Let $\mathcal{O}(\mathrm{Spec}(\mathcal{A}))$ denote the poset of norm-closed ideals of $\mathcal{A}$.

Proposition

$\mathcal{O}(\mathrm{Spec}(\mathcal{A}))$ is a frame. Its points may be identified with the maximal ideals of $\mathcal{A}$.
For any ideal $\mathcal{I} \subseteq \mathcal{A}$ we have

$$\overline{\mathcal{I}} = \bigcap_{\mathcal{I} \subseteq M} M \quad \text{(norm-closure)}.$$

The functor H

We define a functor H

$$\begin{tikzcd} \mathrm{DIV}(\mathcal{A}) \arrow[d, "R\mapsto R\mathcal{A}"'] \arrow[drr, "H"] & & \\ \mathcal{O}(\mathcal{A}) \arrow[rr, "H^*"] & & \mathcal{O}(\mathrm{Spec}(\mathcal{A})) \end{tikzcd} \tag{5}$$

such that

$$H(R) = \overline{R\mathcal{A}} = \bigcap_{R\in M} M\ .$$

$\mathcal{O}(\mathcal{A})$ denotes the frame of $\mathcal{A}$-subsets of $\mathcal{A}$ (a subset of $\mathcal{A}$ that is closed under multiplication by elements of $\mathcal{A}$.)
If $U \preceq R$, then $H(U) \subseteq H(R)$, so that H is indeed a functor.
$H(0) = (0)$, $H(I) = \mathcal{A}$, and $H(RS) = H(R) \cap H(S)$.

The frame map H^*

The suprema extension H^* is given by the norm-closure of the ideal generated by an $\mathcal{A}$-subset $\mathcal{X} \subseteq \mathcal{A}$:

$$H^*(\mathcal{X}) = \overline{(\mathcal{X})} = \bigcap_{\mathcal{X} \subseteq M} M \, .$$

H is flat in the sense that H^* preserves finite infima.
Thus, we have geometric morphisms as follows.

$$\begin{tikzcd}
Sh(\mathrm{Spec}(\mathcal{A})) \arrow[r, tail, "H^* \dashv H_*"] \arrow[d, two heads] & \mathscr{M}/\widehat{I} \arrow[d, two heads, "\text{étale surjection}"] \\
\mathrm{SPEC}(\mathcal{A}) \arrow[r, tail] & \mathscr{M}
\end{tikzcd}$$

GNS representation theory 1

Functionals

A functional on $\mathcal{A}$ is a bounded linear map
$\tau : \mathcal{A} \longrightarrow \mathbb{C}$ (complex numbers).
Let $B(\mathcal{A})$ denote the vector space of functionals on $\mathcal{A}$.
$B(\mathcal{A})$ is a Banach space: $\|\tau\| = \sup\limits_{X \neq 0} \dfrac{|\tau(X)|}{\|X\|}$
$B(\mathcal{A})$ is an $\mathcal{A}$-module: $\tau U(X) = \tau(XU^*)$
$\|\tau U\| \leq \|\tau\| \, \|U\|$ holds (exercise).
$B(\mathcal{A})$ is a Banach $\mathcal{A}$-module.

Positive functional

A functional τ is positive if $0 \leq \tau(A)$ for every positive A.

GNS representation theory 2

Example: sesquilinear form on $\mathcal{A}$

If τ is a functional on $\mathcal{A}$, then $\langle U, V\rangle = \tau(UV^*)$ is a sesquilinear form on $\mathcal{A}$ satisfying

$$\langle UX, V\rangle = \langle U, VX^*\rangle \,.$$

If τ is positive, then $\langle U, V\rangle$ is a semi inner-product, so that $\|U\|_2 = \sqrt{\langle U, U\rangle} = \sqrt{\tau(UU^*)}$ is a seminorm on $\mathcal{A}$.
$\|U\|_2$ is not to be confused with the C^*-algebra norm $\|U\|$ of $\mathcal{A}$.
We have $\|U\|_2 = \|U\|$ just when $\tau(UU^*) = \|UU^*\|$.

Definition: sesquilinear form on an $\mathcal{A}$-module

Let $\mathcal{V}$ be an $\mathcal{A}$-module. A sesquilinear form on $\mathcal{V}$ must satisfy

$$\langle uX, v\rangle = \langle u, vX^*\rangle \,.$$

GNS representation theory 3

The Cauchy-Bunyakovsky-Schwartz Lemma (CBS)

If a sesquilinear form on $\mathcal{V}$ is positive and symmetric, then

$$|\langle u, v\rangle|^2 \leq \langle u, u\rangle\langle v, v\rangle\,.$$

In particular, if τ is positive (whence τ is Hermitian), then

$$|\tau(UV^*)|^2 \leq \tau(UU^*)\tau(VV^*)\,.$$

Annihilators

$$\mathrm{Ann}(\tau) = \{\, V \mid \tau V = 0 \,\}\,,$$

where $\tau V = 0$ means $\tau(UV^*) = 0$ for every $U \in \mathcal{A}$.

Lemma

For any positive τ and V we have $\tau V = 0$ iff $\tau(VV^*) = 0$.

GNS representation theory 4

Proposition

Let τ be a positive functional on $\mathcal{A}$.
Then $U \equiv R \pmod{\mathrm{Ann}(\tau)}$ iff $\tau(UU^*) = \tau(RR^*) = \tau(RU^*)$.

Proof

If $U \equiv R \pmod{\mathrm{Ann}(\tau)}$, then $\tau U = \tau R$, whence
$\tau(UU^*) = \tau(RR^*) = \tau(RU^*)$.
Conversely, if these three are equal, then
$\tau(UU^*) + \tau(RR^*) = \tau(RU^*) + \overline{\tau(RU^*)} = \tau(RU^*) + \tau(UR^*)$,
so that $\tau((U - R)(U^* - R^*)) = 0$, whence $U \equiv R \pmod{\mathrm{Ann}(\tau)}$.

The wide étale subcategory of a positive τ

Let τ be a positive functional on $\mathcal{A}$. Then its wide étale subcategory consists of morphisms $U : R \longrightarrow S$ of $\mathcal{M}$ such that $\tau(UU^*) = \tau(RR^*) = \tau(RU^*)$.

GNS representation theory 5

Module to presheaf

Let $\widehat{B(\mathcal{A})}(R)$ denote

$$\mathcal{A}\text{-MOD}(R\mathcal{A}, B(\mathcal{A})) \cong \{\, \tau \in B(\mathcal{A}) \mid \mathrm{Ann}(R) \subseteq \mathrm{Ann}(\tau) \,\}\,.$$

Transition in $\widehat{B(\mathcal{A})}$ along $U : R \longrightarrow S$ is given by

$$\tau \cdot U = \tau \frac{U}{S} = \tau Y\,;\ U = SY\ .$$

$$\tau \cdot U(X) = \tau Y(X) = \tau(XY^*)\,;\quad \begin{tikzcd} \widehat{R} \arrow[rr, "U"] \arrow[dr, "\tau\cdot U"'] & & \widehat{S} \arrow[dl, "\tau"] \\ & \widehat{B(\mathcal{A})} & \end{tikzcd}$$

$B(\mathcal{A})$ classifies functionals

Let $\mathcal{V}$ be a seminormed $\mathcal{A}$-module.

$$\mathcal{A}\text{-BDD}(\mathcal{V}, B(\mathcal{A})) \underset{\widehat{\delta} \leftarrow\!\shortmid \delta}{\overset{\gamma \mapsto \overline{\gamma}}{\rightleftarrows}} B(\mathcal{V})$$

$$\overline{\gamma}(v) = \overline{\gamma(v)(I)}\ ;\ \widehat{\delta}(v)(U) = \overline{\delta(vU^*)}$$

For instance,
$\overline{\gamma}(vz) = \overline{\gamma(vz)(I)} = \overline{\gamma(v)(\overline{z})} = \overline{\gamma(v)(I)\overline{z}} = \overline{\gamma}(v)z\,.$

$\widehat{B(\mathcal{A})}$ again

Let $\mathcal{V} = R\mathcal{A}$

$$
\begin{tikzcd}
\{\, \tau \in B(\mathcal{A}) \mid \mathrm{Ann}(R) \subseteq \mathrm{Ann}(\tau) \,\} \arrow[r, bend left, "\tau \mapsto \overline{\tau}"] & B(R\mathcal{A}) \arrow[l, bend left, "\widehat{\delta} \leftarrow\!\shortmid \delta"]
\end{tikzcd}
$$

$$\overline{\tau}(RX) = \overline{\tau(X^*)} = \tau^{\dagger}(X)\ ;\ \widehat{\delta}(U) = \overline{\delta(RU^*)}$$

$\overline{\tau}$ is well-defined: if $RX = RY$, then $\tau X = \tau Y$ so that $\tau(X^*) = \tau X(I) = \tau Y(I) = \tau(Y^*)$.
Both sides are $\mathcal{A}$-modules (actually $\mathcal{A}$-algebras): the right side in the usual way (1), and the left side by $\tau \bullet U(X) = \tau(UX)$. Note that $\mathrm{Ann}(\tau) \subseteq \mathrm{Ann}(\tau \bullet U)$. Furthermore, the bijection respects the two actions.

On our way...

We are prepared to analyzed the geometric morphism

$$\begin{tikzcd} \mathscr{M} / \widehat{B(\mathcal{A})} \arrow[d] \\ \mathscr{M} \end{tikzcd}$$

Bibliography 1

[1] B. Blackader.
Operator Algebras: Theory of C^*-Algebras and von Neumann Algebras III, volume 122 of *Encylopedia of Mathematical Sciences*.
Springer, Berline Heidelberg New York, 2006.

[2] J. J. Kohila and V. Rakocevic.
On the norm of idempotents in C^*-algebras.
unpublished.

Thank you